On the Deviatoric Decomposition of the Odroyd-type Viscoelastic Models for Blood Flow Simulations

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Abstract

The flow of blood (and some other biological fluids) can in certain regimes quite accurately be described by viscoelastic models of Oldroyd-type. Of course, in the case of blood flow these models should be properly generalized to account for shear-thinning effects. This has been done e.g. in previous works by Bodnár, Sequeira and co-workers in [1]-[5]. This contribution presents an alternative reformulation of the Oldroyd-type models describing viscoelastic fluids. For simplicity we only use the constant viscosity model to explain the main ideas, however the extension for shear-thinning variable viscosity generalizations is not only possible, but is trivial. The classical formulation is based on the following governing system:

$$\operatorname{div} \boldsymbol{u} = 0 \tag{1}$$

$$\rho \dot{\boldsymbol{u}} = \operatorname{div} \mathbf{T} - \nabla p \tag{2}$$

Here $\boldsymbol{u}$ stands for the velocity vector, ρ is density, p is pressure. The stress tensor is denoted by $\mathbf{T}$. It consists of the Newtonian (solvent) part $\mathbf{T}_s$ and the viscoelastic part $\mathbf{T}_e$, i.e. $\mathbf{T} = \mathbf{T}_s + \mathbf{T}_e$. These two stress components $\mathbf{T}_s$ and $\mathbf{T}_e$ are defined as follows.

$$\mathbf{T}_s = 2\mu_s \mathbf{D} \tag{3}$$

$$\mathbf{T}_e + \lambda \frac{\delta \mathbf{T}_e}{\delta t} = 2\mu_e \mathbf{D} \tag{4}$$

The symbol $\mathbf{D}$ denotes the symmetric part of the velocity gradient. The physical parameters in this model are the solvent and elastic viscosities μ_s, resp. μ_e and the relaxation time λ. The *convected derivative* $\frac{\delta \mathbf{T}_e}{\delta t}$ in the equation (4) can be expanded as:

$$\frac{\partial \mathbf{T}_e}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \mathbf{T}_e = \frac{2\mu_e}{\lambda}\mathbf{D} - \frac{1}{\lambda}\mathbf{T}_e + (\mathbf{W}\mathbf{T}_e - \mathbf{T}_e\mathbf{W}) - a(\mathbf{D}\mathbf{T}_e + \mathbf{T}_e\mathbf{D}) \qquad a \in \langle -1; 1 \rangle \tag{5}$$

It is well known that the Newtonian part of the stress tensor $\mathbf{T}_s$ has only deviatoric part, i.e. its trace is vanishing due to incompressibility (divergence-free constraint). The

spherical part of the stress tensor is in Newtonian case removed to pressure. This behavior is however not shared by the viscoelastic extra stress tensor $\mathbf{T}_e$. In the above mentioned classical formulation the trace of the tensor $\mathbf{T}_e$ is non-vanishing. The aim of this contribution is to reformulate the model in the way that separates the deviatoric and spherical parts of the tensor $\mathbf{T}_e$. It will be shown that it is possible to reformulate the model in the form where a modified governing system will be derived for the deviatoric part of the extra stress and another equation for a new quantity called elastic pressure.

This new system is equivalent to the classical one (in terms of solution), but has some advantages over it. The new formulation allows for easier quantification of fluid-to-wall forcing in fluid-structure interaction problems. This alternative formulation also seems to be more suitable in studying and understanding the high Weissenberg number problem which is one of the most challenging problems of the non-Newtonian fluid mechanics.

Keywords: Oldroyd , blood, viscoelastic, deviatoric, stress tensor.

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