Existence of weak solutions to the Oberbeck-Boussinesq equations

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Abstract

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, and let $0 < T < +\infty$. In $Q := \Omega \times]\,0, T[$ we consider the following system of PDE's:

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div}\big(\nu(\theta)D(\boldsymbol{u})\big) + \nabla p = (1 - \alpha_0 \theta)\boldsymbol{f}, \tag{2}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla\theta - \operatorname{div}\big(\kappa(\theta)\nabla\theta\big) = \nu(\theta)D(\boldsymbol{u}) : D(\boldsymbol{u}) + \alpha_1 \theta \boldsymbol{f} \cdot \boldsymbol{u} + g, \tag{3}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ velocity, p = pressure, θ = temperature, and $D(\boldsymbol{u}) = \big\{D_{ij}(\boldsymbol{u})\big\}$ rate of strain $\big(D_{ij}(\boldsymbol{u}) := \frac{1}{2}(\partial_i u_j + \partial_j u_i),\ D(\boldsymbol{u}) : D(\boldsymbol{u}) := D_{ij}(\boldsymbol{u})D_{ij}(\boldsymbol{u})\big)$. α_0 and α_1 are constants.
We complete (1) – (4) by the following boundary and initial conditions:

$$\left\{ \begin{array}{c} \boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega \times]\,0, T[\,, \\[2ex] \theta = 0 \quad \text{on} \quad \Gamma_0 \times]\,0, T[\,, \quad \kappa(\theta)\dfrac{\partial \theta}{\partial n} = 0 \quad \text{on} \quad \Gamma_1 \times]\,0, T[\,, \end{array} \right. \tag{4}$$

$$\boldsymbol{u} = \boldsymbol{u}_0, \quad \theta = \theta_0 \quad \text{on} \quad \Omega \times \{0\}\,, \tag{5}$$

where $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$.
System (1) – (3) is a variant of the Oberbeck-Boussinesq approximation established in [1]. This system can be also viewed as an approximate model for the motion of heat conducting fluids with small variations of the density which are only due to changes of the temperature. Special cases of (1) – (5) ($\nu \equiv$ const)

have been discussed in [2] and [3].
We prove the existence of a weak solution $\{\boldsymbol{u}, \theta\}$ to (1) – (3) provided that

$$\bullet \quad \big(|\alpha_0| + |\alpha_1|\big)\|\boldsymbol{f}\|_{L^\infty} \quad \text{is small}, \qquad \text{or} \quad \bullet \quad \frac{\alpha_1}{\alpha_0} \geq 1\,.$$

If $\Gamma_0 = \emptyset$, then there holds the following energy equality:

$$e(t) = e(0) + \int_0^t \int_\Omega \big((\alpha_1 - \alpha_0)\theta \boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{f} \cdot \boldsymbol{u} + g\big)\,\mathrm{d}\,x\,\mathrm{d}\,s \quad \text{for a.e. } t\,,$$

where

$$e(t) := \int_\Omega \Big(\frac{1}{2}\big|\boldsymbol{u}(x,t)\big|^2 + \theta(x,t)\Big)\,\mathrm{d}\,x = \text{ total energy}\,.$$

Keywords: Oberbeck-Boussinesq equations, weak solution, energy equality.

References

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